JOURNAL OF DIFFERENTIAL EQUATIONS 11, 284–298 (1972)

On Nonlinear Diffusion Equations

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Received October 21, 1970

1. INTRODUCTION

In this paper we investigate the nonlinear diffusion equation

$$\frac{\partial u(t, x)}{\partial t} = \mathrm{A}u(t, x) + \sum_{i=0}^{\infty} d_i(t, x)\, u^i(t, x) \tag{$**$}$$

in $R \times R^m$, where the differential operator

$$A = a^{ij}(x) \frac{\partial^2}{\partial x_i\, \partial x_j} + b^i(x) \frac{\partial}{\partial x_i} + c(x) \qquad (a^{ij} = a^{ji}) \tag{1}$$

is strictly elliptic in an m-dimensional euclidean space R^m. The coefficients a^{ij}, b^i, and c are assumed to be real-valued infinitely differentiable functions on R^m which satisfy

$$\max \{\max_{|p| \leqslant 2} (\sup_x | D^p a^{ij}(x)|), \max_{|p| \leqslant 1} (\sup_x | D^p b^i(x)|), \sup_x | c(x)|\} = \eta < \infty,$$

where $p = (p_1, \ldots, p_m)$, $D^p = \partial^{|p|}/\partial x_1^{p_1} \cdots \partial x_m^{p_m}$, and $|p| = p_1 + \cdots + p_m$. Under these conditions, K. Yosida has shown [7, pp. 413–418] that the smallest closed extension $\tilde{A}$ in $L_2(R^m)$ of A is the infinitesimal generator of a holomorphic semigroup of bounded operators on $L_2(R^m)$ of class $\{C_0\}$ which we shall denote by $\{\exp(\tilde{A}t) : t \geqslant 0\}$. Formally, solutions of $(**)$ may be written implicitly in terms of $\{\exp(\tilde{A}t) : t \geqslant 0\}$ and the nonlinear term. Our main purpose is to state conditions under which these formal solutions may be interpreted as solutions in the classical sense.

We begin by giving conditions for discussing solutions of the nonlinear differential equation

$$\frac{du(t, \cdot)}{dt} = \tilde{A}u(t, \cdot) + \sum_{i=0}^{\infty} d_i(t, \cdot)\, u^i(t, \cdot) \tag{$*$}$$

* Research for this paper was partially supported by The City University of New York Faculty Research Program through Grant No. 1048.

in $L_2(R^m)$ where $\tilde{A}$ is an unbounded linear operator. In a recent paper [1], the author and C. T. Taam have obtained several conditions under which $\{\exp(\tilde{A}t) : t \geqslant 0\}$ becomes a contraction semigroup in the spaces L_2 and $C(R^m)$. In particular, let c_0 and $c_0{}^*$ denote the least upper bounds, respectively, of the termination coefficients for A and A^*. Then

$$\| \exp(\tilde{A}t)\|_\infty \leqslant e^{c_0 t} \qquad \text{and} \qquad \| \exp(\tilde{A}t)\|_2 \leqslant e^{(c_0+c_0{}^*/2)t}$$

hold for $t \geqslant 0$. The kind of nonlinearity we assume includes the case where $\sum_{i=0}^{\infty} d_i u^i$ is an analytic function of u in some disk of radius r in L_∞ that vanishes at 0. The coefficients $\{d_i\}$ are required to satisfy a Hölder type condition. In Theorem 2.1, we assume $c_0 < 0$, $c_0 + c_0{}^* < 0$, and discuss the existence, uniqueness, and stability of bounded, periodic, almost periodic, and compact solutions of $(*)$. The initial value problem for $(*)$ is considered in Theorem 2.2.

In Section 3, we state additional conditions and justify two ways in which the solutions $u(t, \cdot)$ of $(*)$ may be interpreted as classical solutions of $(**)$. The main problem is to prove that a solution $u(t, \cdot)$ of $(*)$ belongs to the classical domain of the operator A. This is accomplished by using a version of Weyl's Lemma.

Notation. Let R denote the real line and let Ω be an open subset of R^m. By $C^k(\Omega)$, $0 \leqslant k \leqslant \infty$, we denote the set of all bounded complex-valued functions defined in Ω which have bounded continuous partial derivatives of all finite orders up to and including k. When $k = 0$, we omit the superscript. We introduce the space $C^k[R^m]$ of all functions $f \in C^k(R^m)$ such that $\lim_{|x| \to \infty} D^p f(x) = 0$ for $| p | \leqslant k$. Equipped with the uniform norm

$$\| f \|_\infty = \sup\{| f(x)| : x \in R^m\},$$

$C(R^m)$ and $C[R^m]$ are Banach spaces. We shall also use the space $C_0{}^k(R^m)$ of functions in $C^k(R^m)$ with compact support in R^m.

Let I denote any interval in R. For a function f from I to a complex Banach space $(X, \| \ \|_X)$ of complex-valued functions or classes of complex-valued functions, we write

$$\| f \| = \sup\{\| f(t, \cdot)\|_X : t \in I\},$$

whenever it is finite. f is said to be *compact* if $f(I, \cdot)$ is precompact in X. The following spaces of functions from I to X are complex Banach spaces:

(a) The space $(C(I, X), \| \cdot \|)$ of all bounded continuous functions.

(b) The space $(C_\tau(R, X), \| \cdot \|)$ of all periodic continuous functions with period τ.

(c) The space $(CAP(R, X), \|\cdot\|)$ of all continuous almost periodic functions.

(d) The space $(CC(R, X), \|\cdot\|)$ of all continuous compact functions.

For an operator A defined on a subspace of X into a subspace of some space Y, we denote by $D(A)$ the domain of A in X and by $R(A)$ the range of A in Y.

Let $f \in C(R, L_2)$. Then there exists a function $\bar{f}(t, x)$ measurable on the product set $I \times R^m$ such that $f(t) = \bar{f}(t, \cdot)$ for each $t \in I$. For convenience of notation we shall denote this representation by $f(t, x)$. Let $L_\infty(R^m) = L_\infty$ denote the Banach space of all complex-valued Lebesgue measurable functions f with norm

$$\|f\|_\infty = \text{ess sup}\{|f(x)| : x \in R^m\}.$$

No confusion should result from this dual use of the norm notation $\|\cdot\|_\infty$ for both L_∞ and $C(R^m)$. For a function $f \in C(I, L_2)$, we shall denote by $\|f(t, \cdot)\|_\infty$ or $\|f(\cdot, \cdot)\|_\infty$ the essential supremum of f in the missing variable(s) whenever it exists.

2. Bounded, Periodic, Almost Periodic and Compact Solutions

Let $f : I \to L_2(R^m)$ and set $\tau_h f(t, \cdot) = f(t + h, \cdot)$ for t, $t + h \in I$. We shall say that f belongs to the class $H_2{}^q$ in I, $0 < q \leqslant 1$, if for any t, $t + h \in I$,

$$\| \tau_h f(t, \cdot) - f(t, \cdot)\|_2 \leqslant H \mid h \mid^q$$

for some constant H. For $f : I \to L_\infty(R^m)$ or $f : I \to C(R^m)$ and $0 < q \leqslant 1$ we shall say that $f \in H_\infty{}^q$ if for any t, $t + h \in I$,

$$\| \tau_h f(t, \cdot) - f(t, \cdot)\|_\infty \leqslant H \mid h \mid^q$$

for some constant H. Throughout this section and the following we shall often use the above constant H with any function in $H_2{}^q$ or $H_\infty{}^q$, it being understood that the constant H is not necessarily the same at every occurrence.

Let $f \in L_2 \cap L_\infty$. We note that there are constants C and C^* such that

$$\text{(a)} \quad \| \tilde{A}[\exp(\tilde{A}t) - \exp(\tilde{A}s)]\|_2 \leqslant 4C^2 \left(\frac{t-s}{ts}\right), \qquad (0 < s \leqslant t \leqslant 1) \tag{2}$$

$$\text{(b)} \quad \| \tilde{A}[\exp(\tilde{A}t) - \exp(\tilde{A}s)] f \|_\infty \leqslant 4(C^*)^2 \left(\frac{t-s}{ts}\right) \|f\|_\infty, \qquad (0 < s \leqslant t \leqslant 1).$$

These inequalities follow from [1; (7), (22), and Theorem 2] since

$$\tilde{A}[\exp(\tilde{A}t) - \exp(\tilde{A}s)]f = \int_s^t \left[\tilde{A}\exp\left(\tilde{A}\frac{r}{2}\right)\right]^2 f\,dr.$$

We now investigate the ordinary differential equation $(*)$ in L_2 under the following conditions:

(a) $d_i \in C(I, L_\infty) \cap H_\infty{}^q$, $i = 0, 1, \ldots$. (3)

(b) $d_0 \in C(I, L_2) \cap H_2{}^q$.

(c) Let $\delta_i = \| d_i(\cdot, \cdot)\|_\infty$. For some $r > 0$ we have

(i) $\sum_{i=0}^\infty \delta_i r^i < r \mid c_0 \mid$ and (ii) $\sum_{i=1}^\infty i\delta_i r^{i-1} < \min(\mid a_0 \mid, \mid c_0 \mid)$.

(d) For each index i, let H_i denote the Hölder coefficient for d_i in $H_\infty{}^q$. For r in (c) we have

$$\sum_{i=1}^\infty H_i r^i < \infty.$$

(a) $$c_0 < 0,$$ (4)

(b) $$a_0 = \frac{c_0 + c_0{}^*}{2} < 0.$$

We note that when the coefficients $d_i = d_i(t, x)$ are independent of x and continuous in $t \in I$, and if for each $t \in I$ the nonlinear term $\sum_{i=0}^\infty d_i u^i$ is an analytic function of u in some disk of radius $r > 0$ in L_∞ that vanishes at 0, then all conditions in (3) are satisfied when $\mid c_0 \mid$ is sufficiently large.

Definition. A function $u : I \to L_2$ is called a bounded solution of the differential equation $(*)$ if and only if

(a) $u(t, \cdot) \in C(I, L_2)$,

(b) the strong derivative $du(t, \cdot)/dt$ exists and is strongly continuous in I,

(c) $u(t, \cdot) \in D(\tilde{A})$ for each $t \in I$.

(d) $u(t, \cdot)$ satisfies $(*)$ for all $t \in I$.

We seek solutions of $(*)$ which lie in closed sets of the form

$$S(r, I) = \{v \in C(I, L_2) : \| v(\cdot, \cdot)\|_\infty \leqslant r < \infty\}.$$

Theorem 2.1. *Let conditions* (3) *and* (4) *be satisfied with* $I = R$. *Then*

(a) *there is one and only one bounded solution* $u(t, \cdot)$ *of* $(*)$ *in* $S(r, R)$; *in fact,* $\| u(\cdot, \cdot)\|_\infty < r$ *and* $\| u \| \leqslant \| d_0 \|(\mid a_0 \mid - \sum_{i=1}^\infty \delta_i r^{i-1})^{-1}$.

(b) $u \in H_2^q \cap H_\infty^q$.

(c) *$u(t, \cdot)$ is negatively unstable in the sense that for any $a \in R$ and any other strongly continuous function $v(t, \cdot)$ which satisfies $(*)$ a.e. in $(-\infty, a]$ and $\| v(a, \cdot)\|_\infty \leqslant r$ must also satisfy $\| v(t, \cdot)\|_\infty > r$ for infinitely many t without lower bound.*

(d) *$u(t, \cdot)$ is positively asymptotically stable in the sense that there exist two positive numbers δ and ω such that for any $a \in R$ and any other solution $v(t, \cdot) \in C([a, \infty), L_2)$ of $(*)$, each one of the following conditions*

$$\| u(a, \cdot) - v(a, \cdot)\|_\infty \leqslant \delta \quad \textit{and} \quad \| v(t, \cdot)\|_\infty \leqslant r \quad \textit{for} \quad t \in [a, a + \omega]$$

implies that $\| v(t, \cdot)\|_\infty \leqslant r$ for $t \in [a, \infty)$ and $\| u(t, \cdot) - v(t, \cdot)\|_\infty \to 0$ exponentially as $t \uparrow \infty$.

Proof. We define an operator T on $S(r, R)$ by

$$Tv(t, \cdot) = \sum_{i=0}^{\infty} \int_0^\infty \exp(\tilde{A}s)\, d_i(t - s, \cdot)\, v^i(t - s, \cdot)\, ds, \tag{5}$$

where the integral is taken in the sense of Bochner in L_2. The existence of the integral follows using [1, Theorem 2] and (4) from

$$\begin{aligned} \| Tv(t, \cdot)\|_2 &\leqslant \sum_{i=0}^{\infty} \int_0^\infty e^{a_0 s} \| d_i(t - s, \cdot)\, v^i(t - s, \cdot)\|_2\, ds \\ &\leqslant \frac{\| d_0 \|}{| a_0 |} + \frac{\| v \|}{| a_0 |} \sum_{i=1}^{\infty} \delta_i r^{i-1}. \end{aligned}$$

The integral also exists in the Lebesgue sense since, for any $\epsilon > 0$,

$$\begin{aligned} &\left| \sum_{i=0}^{\infty} \int_\epsilon^\infty [\exp(\tilde{A}s)\, d_i(t - s, \cdot)\, v^i(t - s, \cdot)](x)\, ds \right| \\ &\qquad \leqslant \sum_{i=0}^{\infty} \delta_i r^i \int_0^\infty e^{c_0 s}\, ds = \frac{1}{| c_0 |} \sum_{i=0}^{\infty} \delta_i r^i. \end{aligned}$$

Further, $[Tv(t, \cdot)](x) = Tv(t, x)$ a.e. in $R \times R^m$. Indeed, from Hille–Phillips [3, p. 71] we have for any $\beta > 0$

$$\begin{aligned} &\left[\int_0^\beta \exp(\tilde{A}s)\, d_i(t - s, \cdot)\, v^i(t - s, \cdot)\, ds \right](x) \\ &\qquad = \int_0^\beta [\exp(\tilde{A}s)\, d_i(t - s, \cdot)\, v^i(t - s, \cdot)](x)\, ds \end{aligned}$$

a.e. in $R \times R^m$.

Given $\epsilon > 0$, we may choose β and N so large that

$$\| Tv(t, \cdot) - Tv(t', \cdot)\|_2 \leqslant \sum_{i=0}^{N} \int_0^{\beta} e^{a_0 s} \| d_i(t - s, \cdot)\, v^i(t - s, \cdot) - d_i(t' - s, \cdot)\, v^i(t' - s, \cdot)\|_2\, ds + \epsilon.$$

From the uniform continuity of the products $d_i(t - s, \cdot)\, v^i(t - s, \cdot)$ on finite intervals, it follows that $Tv \in C(R, L_2)$. From the hypothesis, $\| Tv(\cdot, \cdot)\|_\infty < r$. Thus, T maps $S(r, R)$ into itself. For $v, w \in S(r, R)$,

$$\begin{aligned} \| Tv(t, \cdot) - Tw(t, \cdot)\|_2 &\leqslant \sum_{i=0}^{\infty} \int_0^{\infty} \| \exp(\tilde{A}s)[d_i(t - s, \cdot)(v^i(t - s, \cdot) - w^i(t - s, \cdot))]\|_2\, ds \\ &< \frac{1}{| a_0 |} \sum_{i=1}^{\infty} i\delta_i r^{i-1} \| v - w \| = \alpha \| v - w \|. \end{aligned}$$

Since $\alpha < 1$, T is a contraction operator. From the contraction mapping principle, there is a unique fixed point $u \in S(r, R)$ which satisfies $\| u(\cdot, \cdot)\|_\infty < r$. Also, from (3) we have

$$\begin{aligned} \| \tau_h u(t, \cdot) - u(t, \cdot)\|_2 \leqslant \int_0^{\infty} e^{a_0 s} \Big\{ &\| \tau_h\, d_0(t - s, \cdot) - d_0(t - s, \cdot)\|_2 \\ &+ \sum_{i=1}^{\infty} [\| \tau_h\, d_i(t - s, \cdot) - d_i(t - s, \cdot)\|_\infty \| \tau_h u^i(t - s, \cdot)\|_2 \\ &+ \| \tau_h u^i(t - s, \cdot) - u^i(t - s, \cdot)\|_2 \| d_i(\cdot, \cdot)\|_\infty] \Big\}\, ds \\ \leqslant \frac{H_0}{| a_0 |} | h |^q &+ \sum_{i=1}^{\infty} \frac{H_i \| u \| r^{i-1}}{| a_0 |} | h |^q + \alpha \| \tau_h u - u \|. \end{aligned}$$

Thus, $u \in H_2{}^q$. Similarly, we find $u \in H_\infty{}^q$ using (3)(c).

Let $0 < \epsilon < \epsilon'$ and set

$$u_\epsilon(t, \cdot) = \int_\epsilon^{\infty} \exp(\tilde{A}s) \left[\sum_{i=0}^{\infty} d_i(t - s, \cdot)\, u^i(t - s, \cdot) \right] ds. \tag{6}$$

Since $\tilde{A} \exp(\tilde{A}\epsilon)$ is a bounded linear operator, from Hille–Phillips [3, p. 83] we have

$$\begin{aligned} &\int_\epsilon^{\infty} \tilde{A} \exp(\tilde{A}s) \left[\sum_{i=0}^{\infty} d_i(t - s, \cdot)\, u^i(t - s, \cdot) \right] ds \\ &\quad = \tilde{A} \exp(\tilde{A}\epsilon) \int_\epsilon^{\infty} \exp(\tilde{A}(s - \epsilon)) \left[\sum_{i=0}^{\infty} d_i(t - s, \cdot)\, u^i(t - s, \cdot) \right] ds \\ &\quad = \tilde{A} u_\epsilon(t, \cdot). \end{aligned}$$

Thus, $u_\epsilon(t,\cdot)\in D(\tilde{A})$ for each $\epsilon>0$. To prove $u(t,\cdot)\in D(\tilde{A})$, since $\tilde{A}$ is a closed linear operator, it suffices to show

$$\left\| \int_\epsilon^{\epsilon'} \tilde{A}\exp(\tilde{A}s)\left[\sum_{i=0}^{\infty} d_i(t-s,\cdot)\, u^i(t-s,\cdot)\right] ds \right\|_2 \to 0$$

as $\epsilon,\epsilon'\to 0$. From [1, (7)] and (4), we have

$$\begin{aligned}
&\left\| \int_\epsilon^{\epsilon'} \tilde{A}\exp(\tilde{A}s)\left[\sum_{i=0}^{\infty} d_i(t-s,\cdot)\, u^i(t-s,\cdot)\right] ds \right\|_2 \\
&\leqslant \int_\epsilon^{\epsilon'} \| s\tilde{A}\exp(\tilde{A}s)\|_2 \left\{ \frac{\| d_0(t-s,\cdot)-d_0(t,\cdot)\|_2}{s} \right. \\
&\quad + \sum_{i=1}^{\infty} \left\{ \| u^i(t-s,\cdot)\|_2 \frac{\| d_i(t-s,\cdot)-d_i(t,\cdot)\|_\infty}{s} \right. \\
&\quad \left.\left. + \| d_i(t,\cdot)\|_\infty \frac{\| u^i(t-s,\cdot)-u^i(t,\cdot)\|_2}{s} \right\}\right\} ds \\
&\quad + \left\| \int_\epsilon^{\epsilon'} \frac{d}{ds}\left\{ \exp(\tilde{A}s)\left[\sum_{i=0}^{\infty} d_i(t,\cdot)\, u^i(t,\cdot)\right]\right\} ds \right\|_2 \\
&\leqslant \left[CH + \sum_{i=1}^{\infty} CH_i \| u \| r^{i-1}\right] \int_\epsilon^{\epsilon'} s^{q-1}\, ds \\
&\quad + \sum_{i=0}^{\infty} \| [\exp(\tilde{A}\epsilon') - \exp(\tilde{A}\epsilon)]\, d_i(t,\cdot)\, u^i(t,\cdot)\|_2
\end{aligned}$$

which tends to zero uniformly for t in a bounded set as $\epsilon,\epsilon'\to 0$. Thus, $\tilde{A}u(t,\cdot) = s-\lim_{\epsilon\downarrow 0}\tilde{A}u(t,\cdot)$ and $\tilde{A}u(t,\cdot)\in C(I,L_2)$ for any bounded interval in I.

We now show that $u(t,\cdot)$ satisfies $(*)$ for all $t\in R$. For $h>0$, we write

$$\begin{aligned}
\frac{u(t+h,\cdot)-u(t,\cdot)}{h} = &\left[\frac{\exp(\tilde{A}h)-I}{h}\right] \\
&\cdot \left\{ \sum_{i=0}^{\infty} \int_{-\infty}^{t} \exp(\tilde{A}(t-s))\, d_i(s,\cdot)\, u^i(s,\cdot)\, ds \right\} \qquad (7) \\
&+ \sum_{i=0}^{\infty} \left\{ \frac{1}{h}\int_t^{t+h} \exp(\tilde{A}(t+h-s))\, d_i(s,\cdot)\, u^i(s,\cdot)\, ds \right\}.
\end{aligned}$$

Letting $h \downarrow 0$, the last expression gives the derivative of the integral while the preceding expression yields $\tilde{A}u(t, \cdot)$ by definition of infinitesimal generator. A similar situation exists for $h < 0$. Thus, $u(t, \cdot)$ satisfies $(*)$. The uniqueness of u follows as in [4, p. 858]. The proof of part (c) follows as in [4, p. 859]. Part (d) follows as in [6, p. 491].

LEMMA 2.1. *If* $0 < q < 1$ *in* (3) *then* $\tilde{A}u(t, \cdot) \in H_2{}^q$.

Proof. We shall write

$$u(t, \cdot) = \int_{-\infty}^{t} \exp(\tilde{A}(t-s)) \left[\sum_{i=0}^{\infty} (d_i(s, \cdot)\, u^i(s, \cdot) - d_i(t, \cdot)\, u^i(t, \cdot)) \right] ds$$
$$+ \int_{-\infty}^{t} \exp(\tilde{A}s) \left[\sum_{i=0}^{\infty} d_i(t, \cdot)\, u^i(t, \cdot) \right] ds.$$

Thus, for $h > 0$

$$\begin{aligned}
&\tilde{A}u(t+h, \cdot) - \tilde{A}u(t, \cdot) \\
&= s - \lim_{\epsilon \downarrow 0} \int_{-\infty}^{t-\epsilon} \tilde{A}[\exp(\tilde{A}(t+h-s)) - \exp(\tilde{A}(t-s))] \\
&\quad \cdot \left\{ \sum_{i=0}^{\infty} (d_i(s, \cdot)\, u^i(s, \cdot) - d_i(t, \cdot)\, u^i(t, \cdot)) \right\} ds \\
&\quad + \int_{-\infty}^{t} \tilde{A} \exp(\tilde{A}(t+h-s)) \left\{ \sum_{i=0}^{\infty} (d_i(t, \cdot)\, u^i(t, \cdot) \right. \\
&\qquad\qquad \left. - d_i(t+h, \cdot)\, u^i(t+h, \cdot)) \right\} ds \\
&\quad + s - \lim_{\epsilon \downarrow 0} \int_{t}^{t+h-\epsilon} \tilde{A} \exp(\tilde{A}(t+h-s)) \left\{ \sum_{i=0}^{\infty} (d_i(s, \cdot)\, u^i(s, \cdot) \right. \\
&\qquad\qquad \left. - d_i(t+h, \cdot)\, u^i(t+h, \cdot)) \right\} ds \\
&\quad + s - \lim_{\epsilon \downarrow 0} \int_{-\infty}^{t+h-\epsilon} \tilde{A} \exp(\tilde{A}(t+h-s)) \left\{ \sum_{i=0}^{\infty} d_i(t+h, \cdot)\, u^i(t+h, \cdot) \right\} ds \\
&\quad - s - \lim_{\epsilon \downarrow 0} \int_{-\infty}^{t-\epsilon} \tilde{A}(\exp(\tilde{A}(t-s)) \left\{ \sum_{i=0}^{\infty} d_i(t, \cdot)\, u^i(t, \cdot) \right\} ds \\
&= J_1 + J_2 + J_3 + J_4 - J_5 \,.
\end{aligned}$$

We may write

$$J_5 = s - \lim_{\epsilon \downarrow 0} \int_{-1/\epsilon}^{t-\epsilon} \tilde{A} \exp(\tilde{A}(t-s)) \left\{ \sum_{i=0}^{\infty} d_i(t, \cdot)\, u^i(t, \cdot) \right\} ds.$$

Indeed fix $\epsilon' > \epsilon$ so small that $t + 1/\epsilon' > 0$. Then from [1, (7)] we have

$$\left\| \int_{-\infty}^{-1/\epsilon} \tilde{A} \exp(\tilde{A}(t - s))\, d_i(t, \cdot)\, u^i(t, \cdot)\, ds \right\|_2$$
$$\leqslant C \| d_i(t, \cdot)\, u^i(t, \cdot)\|_2 \int_{1/\epsilon - 1/\epsilon'}^{\infty} e^{a_0 s}\, ds$$

which tends to zero uniformly for all $t > -1/\epsilon'$ as $\epsilon \downarrow 0$. Thus, upon integrating $J_4 - J_5$ we find $\| J_4 - J_5 \|_2 \leqslant Hh^q$. To estimate J_1 we use (2) and (3) and obtain the bound

$$\int_{-\infty}^{t} \| \tilde{A}[\exp(\tilde{A}(t + h - s)) - \exp(\tilde{A}(t - s))]\|_2$$
$$\cdot \left\| \sum_{i=0}^{\infty} (d_i(s, \cdot)\, u^i(s, \cdot) - d_i(t, \cdot)\, u^i(t, \cdot)) \right\|_2 ds$$
$$\leqslant 4C^2 Hh \int_{-\infty}^{t} (t + h - s)^{-1} (t - s)^{q-1}\, ds = [4C^2 H\pi Csc(q\pi)]\, h^q.$$

Upon integrating, J_2 is bounded by

$$\| \exp(\tilde{A}h)\|_2 \left\| \sum_{i=0}^{\infty} (d_i(t, \cdot)\, u^i(t, \cdot) - d_i(t + h, \cdot)\, u^i(t + h, \cdot)) \right\|_2 \leqslant Hh^q.$$

Also,

$$\| J_3 \|_2 \leqslant HC \int_{t}^{t+h} (t + h - s)^{q-1}\, ds = \frac{1}{q} HC\, h^q.$$

Collecting the above estimates, we obtain the result for $h > 0$. A similar argument holds for $h < 0$.

COROLLARY 2.1. (a) *If the coefficients $d_i(t, \cdot)$ are periodic in t of the same period τ, then the solution $u(t, \cdot)$ of $(*)$ is periodic in t of period τ.*

(b) *If the coefficients $d_i(t, \cdot)$ are almost periodic in t then the solution $u(t, \cdot)$ of $(*)$ is almost periodic in t.*

(c) *If the coefficients $d_i(t, \cdot)$ are compact in L_2 then the solution $u(t, \cdot)$ of $(*)$ is compact.*

Proof. (a) follows immediately upon direct verification that the operator T in (5) maps $S(r, R) \cap C_\tau(R, L_2)$ into itself. From Lemma 2 in [4, p. 854] we obtain that T maps $S(r, R) \cap CAP(R, L_2)$ into itself. For (c), it follows from Theorem 1 in [5, p. 274] that T maps $S(r, R) \cap CC(R, L_2)$ into itself.

THEOREM 2.2. *Let the conditions in* (3) *except* (3)(c) *be satisfied in* $I = [a, \infty)$ *and suppose* a_0 *and* c_0 *in* (4) *are any real numbers. In place of* (3)(c) *we assume that for some* $r > 0$ *we have*

$$\max\left(\sum_{i=0}^{\infty} \delta_i r^{i-1}, \sum_{i=1}^{\infty} i\delta_i r^{i-1}\right) < 1.$$

Then for any $f \in L_2 \cap L_\infty$ *which satisfies* $\| f \|_\infty < r$ *there is a unique bounded solution* $u(t, \cdot)$ *of* $(*)$ *on some interval* $(a, b]$ *in* $S(r, [a, b])$ *which satisfies the initial condition*

$$\lim_{t \downarrow a} \| u(t, \cdot) - f(\cdot) \|_2 = 0.$$

For any $\epsilon > 0$, $u \in H_2{}^q$ *on* $[a + \epsilon, b]$. *Moreover, if* $0 < q < 1$ *in* (3) *then* $\tilde{A}u \in H_2{}^q$ *on* $[a + \epsilon, b]$.

Proof. If $u(t, \cdot)$ is a solution of $(*)$ on $[a, b]$, then [4, p. 873] for $t \in [a, b]$ we have

$$\begin{aligned} u(t, \cdot) &= \exp(\tilde{A}(t - a))\, u(a, \cdot) \\ &\quad + \sum_{i=0}^{\infty} \int_a^t \exp(\tilde{A}(t - s))[d_i(s, \cdot)\, u^i(s, \cdot)]\, ds. \end{aligned} \tag{8}$$

Let $u(a, \cdot) = f(\cdot)$. As in Theorem 2.1, the principle of contraction mapping yields a unique function $u(t, \cdot)$ in $S(r, [a, b])$ satisfying (8) for b chosen such that the following conditions hold:

$$e^{c_0(b-a)} \| f \|_\infty + \sum_{i=0}^{\infty} \delta_i r^i \int_a^b e^{c_0(b-s)}\, ds < r,$$

$$\left(\sum_{i=1}^{\infty} i\delta_i r^{i-1}\right) \min\left\{\int_a^b e^{c_0(b-s)}\, ds, \int_a^b e^{a_0(b-s)}\, ds\right\} < 1.$$

For any $\epsilon > 0$ such that $b - a > \epsilon > 0$ and $t \in (a + \epsilon, b)$, since

$$\frac{d}{dt} \exp(\tilde{A}(t - a)) f = \tilde{A} \exp(\tilde{A}(t - a)) f$$

we have $\exp(\tilde{A}(t - a)) f \in H_2{}^q \cap H_\infty{}^q$ on $[a + \epsilon, b]$. Thus, for t, $t + h \in [a + \epsilon, b]$ we have

$$\begin{aligned} &\| \tau_h u(t, \cdot) - u(t, \cdot) \|_2 \leqslant H \mid h \mid^q \\ &\quad + \sum_{i=0}^{\infty} \int_0^{t+h-a} e^{a_0 s} \| \tau_h(d_i(t - s, \cdot)\, u^i(t - s, \cdot)) - d_i(t - s, \cdot)\, u^i(t - s, \cdot) \|_2\, ds \\ &\quad + \| F(h) - F(0) \|_2 , \end{aligned}$$

where

$$F(h) = \int_{t-a}^{t+h-a} \exp(\tilde{A}s) \left[\sum_{i=0}^{\infty} d_i(t-s,\cdot)\, u^i(t-s,\cdot) \right] ds$$

for $b - t \geqslant h \geqslant a + \epsilon - t$. Since F is continuous, possesses a continuous derivative and both F and its derivative are bounded independently of $t \in (a, b]$ when $h = 0$, we have $F \in H_2{}^q \cap H_\infty{}^q$ on $[a + \epsilon - t, b - t]$ and the Hölder constant H may be chosen independently of $t \in (a, b]$. Thus, $u \in H_2{}^q$ on $(a, b]$. The methods in Theorem 2.1 carry over to show $u(t,\cdot) \in D(\tilde{A})$ and $\tilde{A}u \in C([a+\epsilon, b], L_2)$. Clearly, $u(t,\cdot)$ satisfies the initial condition. The above arguments together with the argument in Lemma 2.1 show that $\tilde{A}u \in H_2{}^q$ on $[a+\epsilon, b]$.

3. Existence of Solutions for the Nonlinear Diffusion Equation (**)

Definition. We say that $u = u(t, x)$ is a solution of (**) in $I \times R^m$ if all the derivatives of u which occur in (**) are continuous functions in $I \times R^m$ and u satisfies (**) at each point of $I \times R^m$.

We require the following additional conditions on the coefficients d_i in (**):

(a) $d_i(t,\cdot) \in C(I, C^2(R^m))$ for $i = 0, 1,\ldots$. (9)

(b) For each $t \in I$, we have

$$\sum_{i=1}^{\infty} \sup\{\| D^p d_i(t,\cdot)\|_\infty : |p| \leqslant 2\}\, r^i < \infty.$$

(c) $D^p d_0(t,\cdot) \in C(I, L_2)$ for $|p| \leqslant 2$.

The above conditions imply that the coefficients d_i are jointly continuous in t and x and twice continuously differentiable in x.

Theorem 3.1. *Let the hypothesis of Theorem* 2.1 *and condition* (9) *be satisfied for* $I = R$. *Then the solution* $u(t,\cdot)$ *of* (*) *in Theorem* 2.1 *taken in the sense of either the Lebesgue or Bochner integral in* (5) *is a solution of* (**). *Moreover, if* $0 < q < 1$ *in* (3), *then* $\tilde{A}u \in H_\infty{}^q$.

Proof. We first show that for $t > 0$, $\exp(\tilde{A}t)$ maps $L_2 \cap C(R^m)$ into $C[R^m]$. By [1, Theorem 2], it suffices to prove this when the coefficient c in A

satisfies a Hölder condition. For $f \in L_2 \cap C(R^m)$, from [1, Theorem 1] we have $\exp(\tilde{A}t) f \in C(R^m)$ and

$$[\exp(\tilde{A}t) f](x) = e^{c_0 t} \int_{R^m - S_\rho} p(t, x, y) f(y)\, dy + e^{c_0 t} \int_{S_\rho} p(t, x, y) f(y)\, dy, \tag{10}$$

where S_ρ is a closed sphere of radius ρ about the origin in R^m. Given $\epsilon > 0$, we may choose ρ so large that $\int_{R^m - S_\rho} |f(y)|^2\, dy < \epsilon^2/4K$ where K is some constant bounding $\int_{R^m} p^2(t, x, y)\, dy$ for fixed $t > 0$. Using Hölder's inequality on the first integral in (10) gives the bound $\epsilon/2$. For $|x| > 4\rho$ we have from [1, (15)]

$$\int_{S_\rho} p(t, x, y) f(y)\, dy \leqslant M t^{-m/2} \| f \|_2 \int_{S_\rho} e^{-(2\alpha/t)|y|^2}\, dy\, e^{-(\alpha/t)|x|^2}.$$

Thus, $\lim_{|x| \to \infty} [\exp(\tilde{A}t) f](x) = 0$.

We may rewrite (6) as

$$u_\epsilon(t, \cdot) = \exp\left(\tilde{A} \frac{\epsilon}{2}\right) \cdot \left\{ \exp\left(\tilde{A} \frac{\epsilon}{2}\right) \int_\epsilon^\infty \exp(\tilde{A}(s - \epsilon)) \left[\sum_{i=0}^\infty (d_i(t - s, \cdot)\, u^i(t - s, \cdot) \right] ds \right\}$$

so that $u_\epsilon(t, \cdot) \in C[R^m]$ for each t. Noting that $u_\epsilon(t, \cdot)$ lies in $C(R, C[R^m])$ we have $u(t, \cdot) \in C(R, C[R^m])$. Therefore, $[u(t, \cdot)](x)$ is jointly continuous in t and x. For small $\beta > 0$, we write

$$u(t, x) = \sum_{i=0}^\infty \int_{t-\beta}^t [\exp(\tilde{A}(t - s))\, d_i(s, \cdot)\, u^i(s, \cdot)](x)\, ds + [\exp(\tilde{A}\beta)\, u(t - \beta, \cdot)](x).$$

For $0 < \beta' < \beta$,

$$\left| \sum_{i=0}^\infty \int_{t-\beta}^{t-\beta'} [\exp(\tilde{A}(t - s))\, d_i(s, \cdot)\, u^i(s, \cdot)](x)\, ds \right| \leqslant \left(\frac{1}{|c_0|} \sum_{i=0}^\infty \delta_i r^i \right) | e^{\beta c_0} - 1 |,$$

which shows that $u(t, x)$ is jointly continuous in t and x. Thus, $u(t, x) = [u(t, \cdot)](x)$ in $R \times R^m$.

Since $\{\exp(\tilde{A}t) : t \geqslant 0\}$ is of class $\{C_0\}$ on $C[R^m]$ we may treat the convergence of $\tilde{A}u_\epsilon(t, \cdot) \in C[R^m]$ to $\tilde{A}u(t, \cdot)$ in the topology of $C[R^m]$, uniformly for t in a bounded interval, as in the case of L_2 convergence. Therefore, $[\tilde{A}u(t, \cdot)](x)$ is jointly continuous in t and x. Moreover, the proof showing $\tilde{A}u \in H_2{}^q$ in Lemma 2.1 is now valid upon replacing the L_2 norm with the norm in $C[R^m]$. Thus, $\tilde{A}u \in H_\infty{}^q$ when $0 < q < 1$.

Using Theorem 2 in [1], we may show that

$$\frac{\partial}{\partial x_k} u_\epsilon(t, \cdot) = \frac{\partial}{\partial x_k} \exp(\tilde{A}\epsilon)\, u(t - \epsilon, \cdot) \in C[R^m].$$

Further, for $0 < \epsilon < \epsilon'$ we have

$$\left\| \int_\epsilon^{\epsilon'} \frac{\partial}{\partial x_k} \exp(\tilde{A}s) \left[\sum_{i=0}^{\infty} d_i(t - s, \cdot)\, u^i(t - s, \cdot) \right] ds \right\|_\infty \leqslant \left(C_1{}^* \sum_{i=0}^{\infty} \delta_i r^i \right) (\epsilon')^{1/2}$$

which tends to zero uniformly for all t as $\epsilon' \downarrow 0$. Therefore,

$$\frac{\partial}{\partial x_k} u(t, \cdot) \in C(R, C[R^m]).$$

In addition,

$$\left\| \tau_h \frac{\partial}{\partial x_k} u(t, \cdot) - \frac{\partial}{\partial x_k} u(t, \cdot) \right\|_\infty \leqslant \int_0^\infty \frac{2^{1/2} C_1{}^* e^{c_0 s/2}}{s^{1/2}}\, ds\, H \mid h \mid^q.$$

Thus, $(\partial/\partial x_k)\, u(t, \cdot) \in H_\infty{}^q$.

Recalling that $u_\epsilon(t, \cdot) \in C^\infty(R^m)$, for $n = 2, 3, \ldots$, we may write

$$\begin{aligned} Au_\epsilon{}^n(t, \cdot) &= nu_\epsilon^{n-1}(t, \cdot)\, Au_\epsilon(t, \cdot) - (n - 1)\, cu_\epsilon{}^n(t, \cdot) \\ &\quad + n(n - 1)\, a^{ij} u_\epsilon^{n-2}(t, \cdot) \left(\frac{\partial}{\partial x_i} u_\epsilon(t, \cdot) \right) \left(\frac{\partial}{\partial x_j} u_\epsilon(t, \cdot) \right). \end{aligned}$$

Thus, we see that for any bounded interval I, $\tilde{A}u^n(t, \cdot) \in C(I, C[R^m])$. Similarly, for $n = 1, 2, \ldots$, we have

$$\begin{aligned} A(d_n(t, \cdot)\, u_\epsilon{}^n(t, \cdot)) &= d_n(t, \cdot)\, Au_\epsilon{}^n(t, \cdot) + u_\epsilon{}^n(t, \cdot)\, Ad_n(t, \cdot) \\ &\quad - cu_\epsilon{}^n(t, \cdot)\, d_n(t, \cdot) + 2a^{ij} \left(\frac{\partial}{\partial x_i} u^n(t, \cdot) \right) \left(\frac{\partial}{\partial x_j} d_n(t, \cdot) \right), \end{aligned}$$

which, letting $\epsilon \downarrow 0$, converges to $\tilde{A}(d_n(t, \cdot)\, u^n(t, \cdot))$ in the topology of $C[R^m]$, uniformly on any bounded interval I. From (9b) since $\tilde{A}$ is closed, we see that $\sum_{i=0}^\infty d_i(t, \cdot)\, u^i(t, \cdot) \in D(\tilde{A})$ and for any bounded interval I,

$$\tilde{A} \sum_{i=0}^{\infty} d_i(t, \cdot)\, u^i(t, \cdot) \in C(I, C[R^m]).$$

We wish to show that $\partial/\partial x_k[\tilde{A}u(t, \cdot)](x)$ exists and is jointly continuous in t and x. We observe these properties hold for $[(\partial/\partial x_k) Au(t, \cdot)](x)$ since for $0 < \epsilon \leqslant \epsilon' \leqslant 1$ we have

$$\left\| \int_\epsilon^{\epsilon'} \frac{\partial}{\partial x_k} \tilde{A} \exp(\tilde{A}s) \left[\sum_{i=0}^{\infty} d_i(t-s, \cdot)\, u^i(t-s, \cdot) \right] ds \right\|_\infty$$

$$= \left\| \int_\epsilon^{\epsilon'} \frac{\partial}{\partial x_k} \exp(\tilde{A}s) \left[\tilde{A} \sum_{i=0}^{\infty} d_i(t-s, \cdot)\, u^i(t-s, \cdot) \right] ds \right\|_\infty$$

$$\leqslant \left\{ \sup_{s\in[0,1]} \left\| \tilde{A} \sum_{i=0}^{\infty} d_i(t-s, \cdot)\, u^i(t-s, \cdot) \right\|_\infty \right\} \int_\epsilon^{\epsilon'} \frac{C_1^*}{s^{1/2}}\, ds$$

which tends to zero as $\epsilon' \downarrow 0$. The result follows since

$$\frac{\partial}{\partial x_k} [Au_\epsilon(t, \cdot)](x) = \left[\frac{\partial}{\partial x_k} Au_\epsilon(t, \cdot) \right] (x).$$

For each t, $[\tilde{A}u(t, \cdot)](x)$ satisfies a Hölder condition in any bounded open set $\Omega \subset R^m$. For any $\phi \in C_0^\infty(\Omega)$, we have by partial integration,

$$\int_\Omega [u(t, \cdot)](x)\, A^*\phi(x)\, dx = \lim_{\epsilon\downarrow 0} \int_\Omega [Au_\epsilon(t, \cdot)](x)\, \phi(x)\, dx$$
$$= \int_\Omega [\tilde{A}u(t, \cdot)](x)\, \phi(x)\, dx.$$

Using a version of Weyl's lemma [2, p. 199], we find that for each t, $[u(t, \cdot)](x)$ coincides a.e. in Ω with a function which is twice Hölder continuously differentiable in $x \in \Omega$. Thus, in fact, $[u(t, \cdot)](x)$ is C^2 in $x \in R^m$ for each t. Therefore, $\tilde{A}u(t, x) = [\tilde{A}u(t, \cdot)](x)$. Since $u(t, \cdot)$ satisfies $(*)$, we have

$$\left[\frac{du(t, \cdot)}{dt} \right] (x) = Au(t, x) + \sum_{i=0}^{\infty} d_i(t, x)\, u^i(t, x) \quad \text{a.e. in } x \in R^m.$$

We observe that the convergence of the difference quotient in (7) holds in the norm in $C[R^m]$, uniformly for t in bounded sets. Thus, $u(t, x)$ is continuously differentiable in t. Hence, $u(t, x)$ is a solution of $(**)$.

With the necessary modifications indicated in Theorem 3.1, we have

THEOREM 3.2. *Let the hypothesis of Theorem* 2.2 *and condition* (9) *be satisfied for* $I = [a, \infty)$. *Then the solution* $u(t, \cdot)$ *of* $(*)$ *in Theorem* 2.2 *taken in the sense of either the Lebesgue or Bochner integral in* (8) *is twice continuously differentiable in* x *and once continuously differentiable in* t *and satisfies* $(**)$ *continuously on* $(a, b] \times R^m$.

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